

## Superconformal vortex strings

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ABSTRACT: We study the low-energy dynamics of semi-classical vortex strings living above Argyres-Douglas superconformal field theories. The worldsheet theory of the string is shown to be a deformation of the  $\mathbf{CP}^N$  model which flows in the infra-red to a superconformal minimal model. The scaling dimensions of chiral primary operators are determined and the dimensions of the associated relevant perturbations on the worldsheet and in the four dimensional bulk are found to agree. The vortex string thereby provides a map between the A-series of  $N=2$  superconformal theories in two and four dimensions.

KEYWORDS: Supersymmetric gauge theory, Conformal and W Symmetry, Solitons Monopoles and Instantons.

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## 1. Introduction

Vortex strings provide an interesting probe of four dimensional quantum field theories, where questions about the strongly coupled gauge dynamics can be answered by studying the solitonic string worldsheet [1, 2].

In this approach one starts with a  $U(N_c)$  gauge theory, coupled to a number  $N_f$  of fundamental scalar fields  $Q$ , where  $N_f \geq N_c$ . In general, the low-energy physics of interest is strongly coupled; let us call it Phase A. To study this system, the theory is first deformed by inducing a vacuum expectation value for  $Q$ . If  $Q$  is made sufficiently large, and the gauge group is fully Higgsed, then this deformed theory will be weakly coupled. We will refer to this weakly coupled system as Phase B.

While Phase B is amenable to semi-classical analysis, it appear to be an unlikely place to understand the strongly coupled dynamics of Phase A. However, Phase B admits vortex strings, stabilized by the winding of  $Q$  in the plane transverse to the string. While the 4d bulk is weakly coupled, the low-energy 2d dynamics of the string is typically strongly coupled, and has been shown to capture information about the original Phase A of the 4d theory. The low-energy modes of interest on the string worldsheet arise from the embedding of the vortex in the  $U(N)$  gauge group, and the resulting worldsheet dynamics is described by some variant of the  $\mathbf{CP}^{N-1}$  sigma-model [3, 4]. In certain systems, the quantum fluctuations of this 2d  $\mathbf{CP}^{N-1}$  sigma-model mirror the underlying fluctuations of the 4d  $U(N)$  non-Abelian gauge theory in Phase A. Indeed, analogies between 2d sigma-models and 4d gauge theories have been studied for over 30 years: the vortex string provides a map between the two.

The programme described above was first implemented in  $\mathcal{N} = 2$  supersymmetric theories, for which the worldsheet theory of the vortex string has  $\mathcal{N} = (2, 2)$  supersymmetry. It was shown in [1, 2], following earlier work of [5, 6], that one may recover the Seiberg-Witten curve [7–10] from the worldsheet. Moreover, the exact BPS quantum spectrum of the 2d worldsheet theory coincides with that of the 4d gauge theory, with the quarks and W-bosons appearing as elementary excitations of the string, while the monopoles, which are necessarily confined in the Higgs phase, appear as kinks on the vortex string [11]. Systems with less supersymmetry were subsequently discussed in [12, 13] where qualitative agreement between the worldsheet and bulk theories was found.

The purpose of this paper is to study a limit in which the vortex worldsheet becomes superconformal. It is well known that there exist special loci on the moduli space of four dimensional  $\mathcal{N} = 2$  gauge theories where particles carrying mutually non-local charges become massless [14–17]. At these “Argyres-Douglas points”, the low-energy physics is described by a strongly interacting superconformal field theory (SCFT). We will examine the worldsheet theory of the vortex string associated to this point: it is given by the  $\mathcal{N} = (2, 2)$  supersymmetric  $\mathbf{CP}^{N-1}$  sigma-model, deformed through the addition of a classical potential. We will see that the parameters of the potential are tuned so that the  $\mathbf{CP}^{N-1}$  model flows to an interacting SCFT which we identify as the  $A_{N-1}$  minimal model.

We compare the scaling dimensions  $D$  of chiral primary operators in the 2d and 4d SCFTs. The spectrum of relevant perturbations in four dimensions splits into two classes: those with  $D < 1$  and those with  $1 \leq D < 2$ . Deformations in the former class are associated to changing the parameters of the theory, while those in the latter class are associated to changing vacuum expectation values (vevs) of fields [15]. We will show that the former descend to chiral primary deformations on the worldsheet where their scaling dimensions in the 2d SCFT coincide with those computed in 4d. In contrast, perturbations in the latter class take us away from the Higgs vacuum and are not seen directly on the worldsheet.

The paper is organized as follows. Section 2 deals with the bulk theory in Phase A. We review the classical four-dimensional gauge theory of interest, identify its superconformal point and compute the scaling dimensions of chiral primary operators. To my knowledge, this particular Argyres-Douglas point on the moduli space has not been previously discussed in the literature although, as we shall see, the resulting SCFT is not new and falls into the standard ADE classification [16, 17]: we will find the  $A_{2N-1}$  4d SCFT appearing in the moduli space of  $U(N)$  gauge theory with  $N$  hypermultiplets. Section 3 deals with the worldsheet. After reviewing how the  $\mathbf{CP}^{N-1}$  model arises as the low-energy dynamics of the vortex string, we identify its superconformal point and show that the dimensions of chiral primary operators coincide with those in the 4d bulk. We further show how motion along the Higgs branch of the four-dimensional theory induces a superpotential on the worldsheet and comment briefly on a novel type of mirror symmetry of finite  $\mathcal{N} = (2, 2)$  theories.

## 2. The bulk theory

Throughout this paper we will study  $\mathcal{N} = 2$  supersymmetric gauge theories in four dimensions, with  $U(N_c)$  gauge group and  $N_f \geq N_c$  fundamental flavors. In terms of  $\mathcal{N} = 1$

superfields, the  $\mathcal{N} = 2$  theory contains a vector multiplet  $W_\alpha$  and a chiral multiplet  $\Phi$ , both in the adjoint of the gauge group.  $W_\alpha$  and  $\Phi$  form the  $\mathcal{N} = 2$  vector multiplet. There are further chiral multiplets  $Q_i$  in the fundamental  $\mathbf{N}_c$  representation, and  $\tilde{Q}_i$  in the  $\bar{\mathbf{N}}_c$  representation, where  $i = 1, \dots, N_f$  is the flavor index.  $Q_i$  and  $\tilde{Q}_i$  form the  $\mathcal{N} = 2$  hypermultiplet. We denote the complex scalar components of  $\Phi$ ,  $Q_i$  and  $\tilde{Q}_i$  by the same letter.

This theory admits semi-classical vortex strings only when it lives in the Higgs phase, in which the gauge group is spontaneously broken by inducing a vacuum expectation value for  $Q$  (referred to as Phase B in the introduction). We may implement this in a manner consistent with  $\mathcal{N} = 2$  supersymmetry by turning on a Fayet-Iliopoulos parameter  $v^2$  for the central  $U(1) \subset U(N_c)$ . The adjoint-valued D-term equation then imposes  $[\Phi, \Phi^\dagger] = 0$ , together with

$$\sum_{i=1}^{N_f} Q_i^a (Q^\dagger)_b^i - (\tilde{Q}^\dagger)_i^a \tilde{Q}_b^i = v^2 \delta_b^a \tag{2.1}$$

with  $a, b = 1, \dots, N_c$  the color indices. Because the left-hand side of (2.1) has rank  $N_f$ , while the right-hand side has rank  $N_c$ , there can be solutions only when  $N_f \geq N_c$  and we restrict to this case. For  $N_f < N_c$  there are no supersymmetric vacua and, more importantly for us, no vortices. The vacuum structure is also dictated by the superpotential, which is fixed by  $\mathcal{N} = 2$  supersymmetry to be of the familiar form  $\mathcal{W} = \sum_i \tilde{Q}_i (\Phi - m_i) Q_i$  with  $m_i$  complex mass parameters. The resulting F-term equations are

$$\sum_{i=1}^{N_f} Q_i^a \tilde{Q}_b^i = 0 \quad , \quad \sum_{b=1}^{N_c} \Phi_b^a Q_i^b = m_i Q_i^a \quad , \quad \sum_{b=1}^{N_c} \tilde{Q}_b^i \Phi_a^b = m_i \tilde{Q}_a^i \tag{2.2}$$

The supersymmetric vacuum states of the theory are given by solutions to (2.1) and (2.2), together with  $[\Phi, \Phi^\dagger] = 0$ . When  $v^2 = 0$ , there is a Coulomb branch of vacua, parameterized by  $\Phi = \text{diag}(\phi_1, \dots, \phi_N)$ . This Coulomb branch is the Phase A referred to in the introduction. In contrast, when  $v^2 \neq 0$ , the Coulomb branch is lifted and  $\Phi$  is forced to take specific values. If the masses  $m_i$  are distinct, there are  $\binom{N_f}{N_c}$  isolated vacua in which  $N_c$  of the  $N_f$  quark fields  $Q$  get an expectation value. Without loss of generality, we choose to work with the vacuum in which the first  $N_c$  flavors turn on,

$$\Phi = \text{diag}(m_1, \dots, m_{N_c}) \quad , \quad Q_i^a = v \delta_i^a \quad , \quad \tilde{Q}_a^i = 0 \tag{2.3}$$

This is Phase B. The spectrum of excitations around this vacuum is gapped. However, as the parameters are varied, new massless fields can appear, sometimes accompanied by new, continuous, branches of vacua. For example, when  $v^2 = 0$ , a Coulomb branch of vacua opens up, parameterized by  $\Phi$ . In contrast, when some subset of the masses coincide, a Higgs branch of vacua opens up, parameterized by gauge invariant combinations of  $Q$  and  $\tilde{Q}$ .

Before discussing the quantum theory, let us pause briefly to examine the pattern of symmetry breaking. As well as the  $U(N_c)$  gauge symmetry, the theory also enjoys an  $SU(N_f)$  flavor symmetry. Both of these are broken spontaneously in the vacuum (2.3) in

way that locks color and flavor rotations together,

$$G \cong \text{U}(N_c) \times \text{SU}(N_f) \xrightarrow{v} H \cong [\text{U}(N_c)_{\text{diag}} \times \text{U}(N_f - N_c)] / \text{U}(1) \quad (2.4)$$

Notice in particular that the central  $\text{U}(1) \subset \text{U}(N_c)$  is broken, providing the topology necessary to support vortex strings. The right-hand side of (2.4) is further, explicitly broken by masses  $m$ , which transform in the adjoint representation of the flavor group. When  $m_i \neq m_j$  for all  $i \neq j$ , only the Cartan subalgebra remains,

$$H \xrightarrow{m} \text{U}(1)^{N_f-1} . \quad (2.5)$$

## 2.1 The superconformal point

In the following section, we will study the quantum dynamics of vortex strings which exist in the classical vacuum (2.3). For now, we wish to study the quantum dynamics in four-dimensions. As explained in the introduction, our interest lies ultimately not in the Phase B vacuum (2.3) — which is weakly coupled when  $v^2$  is sufficiently large — but instead in Phase A with  $v^2 = 0$ . This phase is defined by starting in (2.3), and adiabatically changing  $v^2$  to zero. We wish to ask where on the Coulomb branch we end up. Classically, this vacuum is given by

$$\Phi = \text{diag}(m_1, \dots, m_{N_c}) \quad , \quad Q_i^a = \tilde{Q}_a^i = 0 \quad (2.6)$$

which defines a point on the Coulomb branch. However, this vacuum may receive quantum corrections. In general, the vacuum we want is the point on the Coulomb branch known as the “root of the baryonic Higgs phase”. At this point,  $N_c$  of the  $N_f$  flavors of quarks develop a massless component, ensuring that a FI parameter  $v^2$  may induce a vev for the baryon operator  $B_{1\dots N_c} = \epsilon_{a_1\dots a_{N_c}} Q_1^{a_1} \dots Q_{N_c}^{a_{N_c}} = v^{N_c}$ , without affecting  $\Phi$ .

To determine the correct vacuum on the quantum corrected Coulomb branch, we turn to the Seiberg-Witten curve [7, 8]. For  $N_f < 2N_c$ , the curve is given by<sup>1</sup> [9, 10]

$$y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 - 4\Lambda^{2N_c-N_f} \prod_{i=1}^{N_f} (x - m_i) \quad (2.7)$$

with  $\Lambda$  the strong coupling scale of the gauge theory, given in terms of the 4d gauge coupling  $e^2$ , defined at the RG subtraction point  $\mu$ ,

$$\Lambda^{2N_c-N_f} = \mu^{2N_c-N_f} \exp\left(-\frac{4\pi^2}{e^2(\mu)}\right) \quad (2.8)$$

The presence of  $N_c$  massless quark fields provides a smoking gun in the search for the root of the baryonic Higgs phase, for the curve must develop a suitable degeneracy at this point. Assuming that the first  $N_c$  of the  $N_f$  quarks will become massless, it will prove notationally

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<sup>1</sup>For the  $N_f = 2N_c - 1$  theory, it is customary to shift the masses appearing in the curve by  $m_i \rightarrow m_i + \Lambda/N_c$ .

useful to relabel the excess masses  $\tilde{m}_i = m_{N_c+i}$  with  $i = 1, \dots, N_f - N_c$ . Then the root of the baryonic Higgs phase is given by [6],

$$\prod_{a=1}^{N_c} (x - \phi_a) = \prod_{i=1}^{N_c} (x - m_i) + \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f - N_c} (x - \tilde{m}_i) \quad (2.9)$$

which is to be considered as an equation for  $\phi_a$  with fixed  $m_i$  and  $\tilde{m}_i$ . Note that in the weak coupling regime,  $m_i \gg \Lambda$ , this coincides with the classical vacuum (2.6) while, in the opposite extreme  $m_i = \tilde{m}_i = 0$ , it agrees with the root of the baryonic Higgs branch described in [18]. To see that this is indeed the correct point we can re-examine the curve, which degenerates when (2.9) is obeyed,

$$y^2 = \left( \prod_{i=1}^{N_c} (x - m_i) - \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f - N_c} (x - \tilde{m}_i) \right)^2 \quad (2.10)$$

signalling the presence of the desired  $N$  massless quark hypermultiplets.

For fixed masses  $m_i$  and  $\tilde{m}_i$ , the center of the vortex string is therefore described by the point (2.9) on the Coulomb branch. Our goal now is to tune the masses  $m_i$ , leaving  $\tilde{m}_i$  fixed, such that further states become massless. If these carry mutually nonlocal charges with respect to the quarks, then the resulting four dimensional theory will be superconformal. One finds the maximally degenerate curve is given when the masses  $m_i$  satisfy

$$\prod_{i=1}^{N_c} (x - m_i) = x^{N_c} + \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f - N_c} (x - \tilde{m}_i) \quad (2.11)$$

at which point the curve is simply  $y^2 = x^{2N_c}$ . At this point, magnetic (or dyonic) degrees of freedom become light, joining with the quarks to form an interacting SCFT. There are a number of ways to see that this is indeed the case. For example, a simple criterion was provided in [15] which states that one gets an interacting SCFT if extra particles become massless without opening up new Higgs branches of vacua; one may indeed check that no new vacuum moduli spaces appear when (2.11) is satisfied.

## 2.2 The case $N_f = N_c$

For the remainder of this section, we will focus on the simplest case with  $N_f = N_c \equiv N$  for which all the important elements are present. We will return to the general case of  $N_f > N_c$  in section 3.3. Equation (2.11) defining the superconformal point is now

$$\prod_{i=1}^N (x - m_i) = x^N + \Lambda^N \quad (2.12)$$

which is simply solved by tuning the masses to the critical point,

$$m_k = -\exp(2\pi i k/N) \Lambda \quad , \quad k = 1, \dots, N \quad (2.13)$$

We would now like to identify which SCFT we have found by computing the dimensions of chiral primary operators. This may be achieved by expanding the Seiberg-Witten curve around the singular point [15]. Let us recall how this works:

Our  $\mathcal{N} = 2$  theory has a classical  $U(1)_R$  symmetry that suffers an anomaly: only a  $Z_{2N}$  subgroup survives quantization. This remnant discrete symmetry is itself broken explicitly by the masses. However, at the critical point (2.13), superconformal invariance requires that an accidental  $U(1)_R$  symmetry is restored in the infra-red. This enhanced symmetry is manifest in the curve which, at the singular point, is invariant under  $U(1)_R$  with the charge assignment  $R[y] = NR[x]$ . The dimensions of chiral primary operators in  $\mathcal{N} = 2$  superconformal theories satisfy  $D = 2I + \frac{1}{2}R$ , where  $R$  is the  $U(1)_R$  charge and  $I$  is the  $SU(2)_R$  spin. The chiral primary operators of interest deform the SCFT along the Coulomb branch and have  $I = 0$ . Hence  $D = \frac{1}{2}R$ . Expanding the curve about the superconformal point provides a method to compute the  $R$ -charge, and hence the dimensions, of all chiral primary perturbations. To perform this calculation, it is useful to employ the parametrization,

$$\prod_{a=1}^N (x - \phi_a) = x^N + \sum_{j=1}^N s_j x^{N-j} \quad , \quad \prod_{i=1}^N (x - m_i) = x^N + \sum_{j=2}^N \nu_j x^{N-j} \quad (2.14)$$

Notice that we have set  $\nu_1 = \sum_{i=1}^N m_i = 0$ , which we may always do in a  $U(N_c)$  gauge theory by a suitable shift of  $s_1 = \text{Tr } \Phi$ . The superconformal point (2.13) corresponds to  $s_j = \nu_j = 0$  for  $j = 1, \dots, N - 1$  and  $\nu_N = \Lambda^N$ ,  $s_N = 2\Lambda^N$ . Expanding about this superconformal point, we write  $\nu_j = \hat{\nu}_j$  for  $j = 2, \dots, N - 1$ ;  $s_j = \hat{\nu}_j + \hat{s}_j$  for  $j = 1, \dots, N - 1$ ;  $\nu_N = \Lambda^N + \hat{\nu}_N$  and  $s_N = 2\Lambda^N + \hat{\nu}_N + \hat{s}_N$ . The deformations  $\hat{\nu}_j$  shift both the masses and the expectation values and leave us at the root of the baryonic Higgs phase (2.9). In contrast, the deformations  $\hat{s}_j$  take us away from this locus. Expanding the curve (2.7) for  $N_f = N_c$  around the singularity at  $x = 0$  we have

$$y^2 \approx x^{2N} + 4\Lambda^N \sum_{j=1}^N \hat{s}_j x^{N-j} + 2x^N \left( \sum_{j=2}^N \hat{\nu}_j x^{N-j} \right) + \left( \sum_{j=2}^N \hat{\nu}_j x^{N-j} + \sum_{j=1}^N \hat{s}_j x^{N-j} \right)^2$$

To preserve the the Argyres-Douglas singularity, we must assign relative scaling dimensions to the operators,

$$D[\hat{\nu}_j] = j D[x] \quad \text{and} \quad D[\hat{s}_j] = (N + j) D[x] \quad . \quad (2.15)$$

It remains to determine the dimension of  $x$  itself, and hence the overall normalization. This is fixed using the fact that BPS masses are obtained by integrating the Seiberg-Witten 1-form  $\lambda_{\text{SW}}$  around closed cycles  $\alpha^a$  of the curve. The 1-form is given by

$$\lambda_{\text{SW}} = \frac{1}{2\pi i} \frac{\partial P(x)}{\partial x} \frac{x dx}{y} \quad , \quad P(x) = \prod_{a=1}^N (x - \phi_a) \quad (2.16)$$

The dual scalar  $\phi_D^a$ , which necessarily has dimension  $D[\phi_D^a] = 1$ , is then obtained by the contour integral  $\partial\phi_D^a/\partial s_b = \oint_{\alpha_a} \partial\lambda_{\text{SW}}/\partial s_b$ . The upshot of this is that the dimensions are

constrained to obey  $D[\hat{s}_j] + (N - j + 1) D[x] - D[y] = 1$ , from which we learn the spectrum of relevant perturbations of the SCFT,

$$\begin{aligned} D[\hat{\nu}_j] &= \frac{j}{N+1} & j = 2, \dots, N \\ D[\hat{s}_j] &= \frac{N+j}{N+1} & j = 1, \dots, N \end{aligned} \tag{2.17}$$

The deformations with dimensions  $D < 1$  are associated to varying the mass parameters of the theory and leave us at the root of the baryonic Higgs phase where vortices exist. As we will see shortly, these deformations manifest themselves on the string worldsheet. In contrast, deformations with dimension  $D \geq 1$  involve only a variation of field expectation values, and take us away from the root of the baryonic Higgs phase where no vortex strings live. The two types of deformations are analogous to the familiar non-normalizable and normalizable perturbations in AdS/CFT. As explained in [15], it is a general feature of 4d  $\mathcal{N} = 2$  SCFTs that these two types of relevant parameters come in pairs satisfying

$$D[\hat{\nu}_j] + D[\hat{s}_{N-j+2}] = 2, \tag{2.18}$$

The mass parameters  $m_i$  are associated to the  $U(1)^{N-1}$  flavor symmetry (2.5): once these symmetries are weakly gauged, the masses appear as background expectation values. The fact that  $D[\hat{\nu}_j^{1/j}] \neq 1$  implies that this flavor symmetry does not act in the SCFT, but rather couples, after weak gauging, through irrelevant interactions. Giving an expectation value to turn on the masses then deforms the SCFT by a relevant operator which, from the pairing (2.18), takes the form

$$\delta\mathcal{L} = \sum_{j=2}^N \nu_j \int d^4\theta S_{N-j+2}. \tag{2.19}$$

where  $S_j$  is the  $\mathcal{N} = 2$  superfield containing  $s_j$  as its lowest component and  $\int d^4\theta$  denotes integration over one half of  $\mathcal{N} = 2$  superspace. In contrast, the dimension of  $D[s_1] = 1$  implies that the associated singlet mass  $\text{Tr } \Phi$  couples to a current — identified with  $U(1)_B$  in the  $SU(N_c)$  theory — which gives rise to conserved charges within the SCFT.

We note in passing that our SCFT lies at a different point than those usually discussed in the literature. For example, in [16, 17] one sets all the masses equal,  $m_i = m$ , and subsequently adjusts  $m$  to find further massless particles. Here, however, we have set  $\sum m_i = 0$  and sought the superconformal point lying at the root of the Higgs phase (2.9). Nonetheless, the spectrum of chiral primary operators that we have found falls within the categorization presented in [16, 17]; indeed, the SCFT at the root of the  $U(N)$  baryonic Higgs phase is the same as the one within the pure  $SU(2N)$  super Yang-Mills theory. This is the  $A_{2N-1}$  series of 4d  $\mathcal{N} = 2$  SCFTs.

### 3. The worldsheet theory

In this section we return to Phase B, described by the classical vacuum (2.3), in which the gauge group is fully Higgsed by the expectation value of  $Q$ . We will construct an infinite,



straight vortex string in this background and study its low-energy dynamics. We will show that precisely when the masses are tuned to the Argyres-Douglas point, the worldsheet theory will also flow to a SCFT which we identify as the  $\mathcal{N} = (2, 2)$  minimal model.

In the Higgs vacuum, the topology  $\Pi_1(G/H) \cong \mathbf{Z}$  of the symmetry breaking described in (2.4) supports vortex strings, stabilized by the phase of  $Q_a^i$  winding in the plane transverse to the string. Straight, infinite vortex strings stretched in, say, the  $x^3$  direction are BPS objects described by solutions to the classical non-Abelian Bogomolnyi equations,

$$\mathcal{D}_1 Q_i = i\mathcal{D}_2 Q_i \quad , \quad F_{12} = e^2 \left( \sum_i Q_i Q_i^\dagger - v^2 \right) \quad (3.1)$$

Here  $e^2$  is the gauge coupling constant. Both  $\Phi$  and  $\tilde{Q}$  remain in their classical vacuum state (2.3) in the vortex solution. Equations (3.1) are the non-Abelian generalization of the vortex equations appearing in the Abelian Higgs model. The solutions describe strings of tension  $T = 2\pi v^2$  and width<sup>2</sup>  $L = 1/ev$ . As we now review, the non-Abelian embedding endows the vortex with interesting dynamics.

### 3.1 The classical worldsheet: $N_f = N_c$

The low-energy dynamics of vortex strings always includes two Goldstone modes associated to their transverse fluctuations. More important for us will be further massless (or light) modes on the worldsheet that arise from the embedding of the vortex in the non-Abelian gauge group. In this section we review the internal modes of the vortex, restricting to the  $N_f = N_c \equiv N$  theory. We will return to the general case of  $N_f > N_c$  at the end of the paper. A more complete review of these solitons can be found in [19].

We start by describing the vortex worldsheet dynamics when the masses  $m_i = 0$ . Suppose we have a solution  $(q, a)$  to the Abelian U(1) vortex equations. Then we may always construct a solution to the non-Abelian vortex equations by an embedding in the upper-left-hand corner,

$$Q_i^a = \begin{pmatrix} q & & \\ & \ddots & \\ & & v \end{pmatrix} \quad , \quad A_b^a = \begin{pmatrix} a & & \\ & \ddots & \\ & & 0 \end{pmatrix} \quad (3.2)$$

The vacuum state of the 4d theory has a surviving  $SU(N)_{\text{diag}}$  symmetry, which is the diagonal combination of a gauge and flavor rotation. This acts on the solution (3.2) as  $Q \rightarrow UQU^\dagger$  and  $A \rightarrow UAU^\dagger - i(\partial U)U^\dagger$  to provide further Goldstone modes on the worldsheet. Dividing by the stabilizing group, the internal low-energy dynamics of the vortex is described by a  $d = 1 + 1$  sigma-model with target space [3, 4]

$$SU(N)_{\text{diag}} / SU(N-1) \times U(1) \cong \mathbf{CP}^{N-1} \quad (3.3)$$

The vortex is 1/2-BPS in the  $\mathcal{N} = 2$  4d gauge theory, ensuring that 2d worldsheet theory has  $\mathcal{N} = (2, 2)$  supersymmetry, with the fermi zero modes of the string providing the worldsheet superpartners.

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<sup>2</sup>Throughout this paper we have not distinguished the four-dimensional U(1) gauge coupling from the  $SU(N_c)$  gauge coupling. The width  $L$  of the vortex string is determined by the Abelian coupling constant.

There is a way to write the supersymmetric  $\mathbf{CP}^{N-1}$  sigma-model in terms of an  $\mathcal{N} = (2, 2)$  supersymmetric U(1) gauge theory that will prove useful in solving for the quantum dynamics [20–22]. Consider an auxiliary U(1) field strength on the worldsheet, living in a twisted chiral multiplet  $\Sigma$ , whose lowest component we denote as  $\sigma$ . The gauge field couples to  $N$  chiral multiplets  $\Psi_i$ , each with charge +1. The lowest components of  $\Psi_i$  will be denoted as  $\psi_i$  and play the role of homogeneous coordinates on  $\mathbf{CP}^{N-1}$ . The potential energy of the worldsheet theory is a sum of F-terms and the D-term

$$V_{2d} = \sum_{i=1}^N |\sigma|^2 |\psi_i|^2 + \frac{g^2}{2} \left( \sum_{i=1}^N |\psi_i|^2 - r \right)^2 \tag{3.4}$$

Here  $g^2$  is a gauge coupling on the worldsheet which is irrelevant for the infra-red quantum dynamics at energies  $E \ll g$ . At low-energies, the D-term restricts to  $\sum_i |\psi_i|^2 = r$ . After dividing by U(1) gauge transformations  $\psi_i \rightarrow e^{i\alpha} \psi_i$ , the gauge theory reduces to the sigma-model with target space  $\mathbf{CP}^{N-1}$ . The worldsheet FI parameter  $r$  determines the size of the  $\mathbf{CP}^{N-1}$  target space and, for the vortex moduli space, is given in terms of the 4d gauge coupling [3],

$$r = \frac{2\pi}{e^2} \tag{3.5}$$

The 4d theta angle also descends to a 2d theta angle on the worldsheet [2, 13].

So far we have discussed the theory with vanishing masses  $m_i = 0$ . How do non-zero masses change the worldsheet dynamics of the vortex string? The answer was given in [11, 1, 2]. The masses break the surviving symmetry group  $SU(N)_{\text{diag}} \rightarrow U(1)^{N-1}$  and the associated worldsheet Goldstone modes are lifted. Of the  $\mathbf{CP}^{N-1}$  moduli space of solutions, only  $N$  isolated solutions remain. These correspond to the Abelian vortex  $(q, a)$  embedded in the  $N$  different diagonal elements of  $Q$  and  $A$ . From the perspective of the worldsheet theory, the complex masses  $m_i$  in 4d can be shown to induce twisted masses  $m_i$  [23] in 2d, so that the worldsheet potential energy reads

$$V_{2d} = \sum_{i=1}^N |\sigma - m_i|^2 |\psi_i|^2 + \frac{g^2}{2} \left( \sum_{i=1}^N |\psi_i|^2 - r \right)^2 \tag{3.6}$$

As anticipated, the  $\mathbf{CP}^{N-1}$  target space is lifted, leaving behind  $N$  isolated vacua of the vortex worldsheet given by  $\sigma = m_i$  and  $|\psi_j|^2 = r \delta_{ij}$ , for  $i = 1 \dots, N$ . Kinks in the vortex string, interpolating between these different worldsheet vacua, are confined 4d monopoles [11].

### 3.2 The superconformal worldsheet: $N_f = N_c$

In summary, the classical low-energy dynamics of the vortex string in the theory with  $N_f = N_c$  flavors is described by the  $\mathcal{N} = (2, 2)$   $\mathbf{CP}^{N_c-1}$  sigma model, with the 4d masses  $m_i$  inducing a classical potential over the target space. In this section we study the quantum dynamics of this theory. Typically, the  $\mathbf{CP}^{N-1}$  sigma-model has a mass gap. However, we will show that once the masses are tuned so that the 4d theory lies at the Argyres-Douglas

point, the  $\mathbf{CP}^{N-1}$  sigma model flows to an interacting SCFT which we identify as the  $A_{N-1}$  minimal model.

The quantum effective action for the mass deformed  $\mathbf{CP}^{N-1}$  model was studied in [23, 5, 24]. Following [22], one integrates out the charged 2d chiral multiplets  $\Psi_a$  to find an effective twisted superpotential  $\tilde{\mathcal{W}}$  for the field strength  $\Sigma$ ,

$$\tilde{\mathcal{W}} = -\frac{1}{2\pi} \sum_{i=1}^N (\Sigma - m_i) \left[ \log \left( \frac{\Sigma - m_i}{\mu} \right) - 1 \right] - t\Sigma \tag{3.7}$$

where  $\mu$  is the RG subtraction point. The 2d complexified FI parameter  $t = r + i\theta$  runs under RG flow and is exchanged for the invariant dynamical scale  $\Lambda = \mu \exp(-2\pi t/N)$ . Because the 2d FI parameter is related to the 4d gauge coupling through  $r = 2\pi/e^2$ , the strong coupling scale  $\Lambda$  on the worldsheet coincides with the 4d strong coupling scale (2.8).

The worldsheet theory has  $N$  vacuum states, given by the critical points of  $\tilde{\mathcal{W}}$ ,

$$\prod_{i=1}^{N_c} (\sigma - m_i) - \Lambda^N = 0 \tag{3.8}$$

For large mass differences, so  $|m_i - m_j| \gg |\Lambda|$ , these vacua coincide with the classical vacua  $\sigma = m_i$ . In the opposite regime,  $m_i = 0$ , the  $N$  vacua descend to the strong coupling scale  $\sigma = \omega\Lambda$ , where  $\omega^N = 1$ . Comparing the Seiberg-Witten curve at the root of the baryonic Higgs phase (2.10) to the twisted superpotential (3.7), we see that, for  $N_f = N_c$ , the former may be written as  $y^2 = (\partial\tilde{\mathcal{W}}(x)/\partial x)^2$ . This is the statement that the worldsheet of the vortex string captures the Seiberg-Witten curve.

Let us now tune the masses so that the 4d theory sits at the Argyres-Douglas point identified in the previous section. Following (2.13), we set  $m_k = -\exp(2\pi ik/N)\Lambda$  and examine the consequences for the worldsheet. The importance of this point was stressed in [5, 24]. The  $N$  critical points (3.8) merge at  $\sigma = 0$ , ensuring that the kinks interpolating between different vacuum states become massless. This reflects the behavior of monopoles in the underlying Argyres-Douglas point because, in the Higgs phase, monopoles are confined, trapped to live on the vortex string where they appear as kinks. It was conjectured in [24] that the 2d theory becomes an interacting SCFT at the point (2.13). To see that this is indeed the case, we expand the twisted superpotential (3.7) at this point for small  $\sigma/\Lambda$  to find

$$\tilde{\mathcal{W}} = c_0 \frac{\Sigma^{N+1}}{\Lambda^N} + \dots \tag{3.9}$$

where  $c_0$  is an overall normalization and  $\dots$  refer to irrelevant operators. We see that the familiar logarithms of the  $\mathbf{CP}^{N-1}$  sigma-model are replaced by a polynomial Landau-Ginzburg model. The Kähler potential of the theory is unknown at this strongly coupled point. However, this is unimportant because, while the superpotential is protected by non-renormalization theorems, the Kähler potential is expected to adjust itself under RG flow so that the theory flows to an interacting  $\mathcal{N} = (2, 2)$  SCFT which is identified with the  $A_{N-1}$  minimal model [25–27]. The central charge of this 2d SCFT is

$$c = 3 - \frac{6}{N+1} . \tag{3.10}$$

Representation theory of the  $\mathcal{N} = (2, 2)$  superconformal algebra relates the dimension  $D$  of chiral primary operators to the charge  $R$  under the  $U(1)_R$  symmetry:  $D = \frac{1}{2}R$ . The  $\mathbf{CP}^{N-1}$  sigma-model has a classical  $U(1)_R$  symmetry which is anomalous in the quantum theory, with only a  $\mathbf{Z}_{2N}$  subgroup surviving. This surviving discrete group is further broken explicitly by generic twisted masses  $m_i$ . However, at the critical point in parameter space, where the theory is governed by (3.9), an accidental  $U(1)_R$  symmetry is restored in the infra-red, as required by superconformal invariance. This mirrors the story for the  $U(1)_R$  symmetry in four dimensions; indeed, the worldsheet  $U(1)_R$  is inherited from the 4d  $U(1)_R$ .

Since the twisted superpotential necessarily has R-charge 2, the R-charge of the twisted chiral multiplet  $\Sigma$  is given by  $R[\Sigma] = 2/(N + 1)$ . The spectrum of chiral primary operators therefore have dimensions  $D_j = j/(N + 1)$  where  $j = 1, \dots, N - 1$ . (The addition of the operator  $\Sigma^N$  with  $j = N$  is redundant since it may be absorbed by a constant shift of  $\Sigma$ ). We may identify each of these relevant deformations in terms of the mass parameters  $m_i$ . To do this, rewrite

$$\prod_{i=1}^N (\sigma - m_i) = \sigma^N + \sum_{j=2}^{N-1} \nu_j \sigma^{N-j} \tag{3.11}$$

where there is no  $\nu_1$  since we have chosen  $\sum_{i=1}^N m_i = 0$ . The conformal point (2.13) in parameter space corresponds to  $\nu_j = 0$  for  $j = 1, \dots, N - 1$  and  $\nu_N = \Lambda^N$ . We expand the twisted superpotential (3.7) about this point, writing  $\nu_j = \hat{\nu}_j$  for  $j = 1, \dots, N - 1$  and  $\nu_N = \Lambda_N + \hat{\nu}_N$ , to find

$$\delta\tilde{\mathcal{W}} = \sum_{j=2}^N c_j \hat{\nu}_j \Sigma^{N-j+1} + \dots \tag{3.12}$$

which is to be compared to (2.19), giving the map between bulk and worldsheet chiral operators:  $\Sigma^{N-j+1} \leftrightarrow S_{N-j+2}$ . The dimensions of the relevant perturbations are

$$D[\hat{\nu}_j] = \frac{j}{N+1} \quad j = 2, \dots, N \tag{3.13}$$

in agreement with the four-dimensional result (2.17).

The vortex string thus provides a map between the  $A_{2N-1}$  series of 4d  $\mathcal{N} = 2$  SCFTs, and the  $A_{N-1}$  series of 2d  $\mathcal{N} = (2, 2)$  SCFTs. Although only one half of the 4d relevant operators are realized on the worldsheet (those that leave us at the root of the baryonic Higgs branch), the general feature (2.18) of 4d SCFTs ensures that we can reconstruct the full spectrum of relevant operators from the worldsheet.

Relationships between 4d SCFTs and 2d minimal models have been described previously. In particular, the spectrum of BPS states in the vicinity of an Argyres-Douglas point was shown to bear many similarities to massive deformations of 2d SCFTs [28]. The vortex string provides a rationale for this correspondence, with the 4d BPS states mapping to the 2d BPS states.

### 3.3 Generalization to $N_f > N_c$

So far we have examined the superconformal point only for  $N_f = N_c$ . We now briefly discuss the generalization to  $N_c < N_f \leq 2N_c - 1$  flavors. For distinct masses, there are  $\binom{N_f}{N_c}$  different roots of the baryonic Higgs phase. We choose to work with the root which classically corresponds to the vacuum  $\Phi = \text{diag}(m_1, \dots, m_{N_c})$  and, as in section 2, we relabel the  $(N_f - N_c)$  remaining masses as  $\tilde{m}_i = m_{N_c+i}$ .

It is a simple exercise to expand the curve (2.10) about the superconformal point (2.11) to extract the dimensions of chiral primary operators in the four dimensional SCFT with  $N_f > N_c$ . For generic non-zero masses  $\tilde{m}_i$  one finds that the singularity is unaltered, corresponding once again to a SCFT with scaling dimensions (2.17). The excess masses  $\tilde{m}_i$  in this case are irrelevant deformations. However, this changes when some of the masses  $\tilde{m}_i$  vanish. In this situation the singularity is partially resolved. Consider the extreme case  $\tilde{m}_i = 0$  for all  $i = 1, \dots, N_f - N_c$ . Expanding the curve about the superconformal point (2.11) now gives,

$$y^2 \approx x^{2N_c} + 4\Lambda^{2N_c - N_f} x^{N_f - N_c} \sum_{j=1}^{N_c} \hat{s}_j x^{N_c - j} + 2x^{N_c} \left( \sum_{j=2}^{N_c} \hat{\nu}_j x^{N_c - j} \right) + \left( \sum_{j=2}^{N_c} \hat{\nu}_j x^{N_c - j} \right)^2 + \dots$$

where  $\dots$  are irrelevant terms. The relative scaling dimensions of the various perturbations are now given by  $D[\nu_j] = j[x]$  and  $D[s_j] = (2N_c - N_f + j)[x]$ . The overall normalization remains as before, giving us the dimensions

$$D[\hat{\nu}_j] = \frac{j}{2N_c - N_f + 1} \quad , \quad D[\hat{s}_j] = \frac{2N_c - N_f + j}{2N_c - N_f + 1} \quad (3.14)$$

We will now show how this behavior is captured by the worldsheet. Vortex strings in the theory with  $N_f > N_c$  have a rather different property from those in the  $N_f = N_c$  theory: their scale size is a collective coordinate. (See [29] for a review). An effective dynamics for the string was proposed in [3]. It is an  $\mathcal{N} = (2, 2)$  supersymmetric  $U(1)$  gauge theory with  $N_c$  chiral multiplets  $\Psi_i$  of charge  $+1$  and twisted mass  $m_i$ . There are a further  $(N_f - N_c)$  chiral multiplets  $\tilde{\Psi}_j$  of charge  $-1$  and twisted mass  $\tilde{m}_j$ . The scalar potential on the worldsheet is given by

$$V_{2d} = \sum_{i=1}^{N_c} |\sigma - m_i|^2 |\psi_i|^2 + \sum_{i=1}^{N_f - N_c} |\sigma - \tilde{m}_i|^2 |\tilde{\psi}_i|^2 + \frac{g^2}{2} \left( \sum_{i=1}^{N_c} |\psi_i|^2 - \sum_{j=1}^{N_f - N_f} |\tilde{\psi}_j|^2 - r \right)^2$$

When the masses vanish,  $m_i = \tilde{m}_i = 0$ , the extra modes  $\tilde{\psi}_i$  provide the worldsheet theory with a non-compact moduli space of vacua. This non-compact direction corresponds to the scaling mode of the vortex. For generic values of the masses, this worldsheet theory was shown to share its BPS spectrum with the 4d theory in which the vortex lives [6, 2]. Here we examine this theory at the superconformal point.

Integrating out the chiral multiplets, the effective worldsheet theory is governed by the twisted superpotential,

$$\tilde{\mathcal{W}} = -\frac{1}{2\pi} \sum_{i=1}^{N_c} (\Sigma - m_i) \left[ \log \left( \frac{\Sigma - m_i}{\mu} \right) - 1 \right] + \frac{1}{2\pi} \sum_{j=1}^{N_f - N_c} (\Sigma - \tilde{m}_j) \left[ \log \left( \frac{\Sigma - \tilde{m}_j}{\mu} \right) - 1 \right] - t\Sigma$$

The superconformal point on the worldsheet occurs when all critical points coincide,

$$\prod_{i=1}^{N_c} (\sigma - m_i) = \sigma^{N_c} + \Lambda^{2N_c - N_f} \prod_{j=1}^{N_f - N_c} (\sigma - \tilde{m}_j) \quad (3.15)$$

which is to be understood as an equation for the masses  $m_i$  for fixed  $\tilde{m}_j$ . Notice that, as expected, the equation coincides with the four dimensional superconformal point defined in (2.11). The nature of the worldsheet SCFT depends on the masses  $\tilde{m}_i$ . For  $\tilde{m}_i \neq 0$ , expanding the twisted superpotential about the superconformal point gives

$$\tilde{\mathcal{W}} \sim \frac{\Sigma^{N+1}}{\Lambda^{2N_c - N_f} \prod_j \tilde{m}_j}, \quad (3.16)$$

which we recognize once again as the  $A_{N-1} \mathcal{N} = (2, 2)$  SCFT. However, when  $\tilde{m}_j = 0$  for some  $j$ , the nature of the superconformal point changes. On a technical level this occurs because we may no longer expand  $\tilde{W}$  in  $\sigma/\tilde{m}_j$ . Consider the extreme case when  $\tilde{m}_j = 0$  for all  $j = 1, \dots, N_f - N_c$ . It is simple to repeat the computation above to find worldsheet superpotential

$$\tilde{\mathcal{W}} \sim \frac{\Sigma^{2N_c - N_f + 1}}{\Lambda^{2N_c - N_f}} \quad (3.17)$$

corresponding to a reduced  $A_{2N_c - N_f - 1}$  SCFT. The dimensions of relevant perturbations are now given by

$$D[\hat{\nu}_j] = \frac{j}{2N_c - N_f + 1} \quad (3.18)$$

in agreement with the four dimensional theory (3.14). Before moving on, we pause to note that the validity our starting worldsheet theory is in some doubt in the case  $\tilde{m}_j = 0$ . A classical infra-red divergence means that the scaling modes of the vortex string are non-normalizable [30, 31], a fact that is not obviously captured in the worldsheet theory described above. For this reason, one might expect the worldsheet theory to be valid only when  $\tilde{m}_i \neq 0$ , which ensures that the the infra-red divergence is rendered finite [32]. The result (3.18) shows that, at the superconformal point, the proposed worldsheet theory dynamically freezes the scaling modes when  $\tilde{m}_i = 0$ . This, coupled with the resulting agreement with the four-dimensional SCFT, suggests that the worldsheet theory continues to capture the quantum physics of the vortex string even when  $\tilde{m}_i = 0$ .

### 3.4 Moving on the Higgs branch

When  $m_i = \tilde{m}_i = 0$ , the four dimensional theory with  $N_f > N_c$  has a Higgs branch of vacua of complex dimension  $N_c(N_f - N_c)$ . A gauge invariant description of this branch is provided by expectation values for the meson and baryon operators

$$M_i^j = \tilde{Q}_a^j Q_i^a \quad , \quad B_{i_1 \dots i_{N_c}} = \epsilon_{a_1 \dots a_{N_c}} Q_{i_1}^{a_1} \dots Q_{i_{N_c}}^{a_{N_c}} \quad , \quad \tilde{B}^{i_1 \dots i_{N_c}} = \epsilon^{a_1 \dots a_{N_c}} \tilde{Q}_{a_1}^{i_1} \dots \tilde{Q}_{a_{N_c}}^{i_{N_c}}$$

These are not all independent, but satisfy a number of polynomial relations which must be imposed, together with the D-term (2.1) and F-term equations (2.2), to describe the Higgs

branch in the gauge invariant fashion — see [18] for more details. The presence of the FI parameter in the D-term (2.1) deforms, but does not lift, the Higgs branch.<sup>3</sup>

Let us return momentarily to a description of the Higgs branch in terms of the gauge non-invariant fields. A combination of gauge and flavor rotations allows the general solution to the D-term (2.1) and F-term (2.2) equations to be put in the form [18]

$$Q_i^a = q^a \delta_i^a \quad , \quad \tilde{Q}_a^i = \tilde{q}_a \delta_{a+N_c}^i \quad \text{no sum on } a \quad (3.19)$$

subject to  $|q^a|^2 - |\tilde{q}_a|^2 = v^2$  for each  $a = 1, \dots, N_c$ . Note that  $\tilde{q}_a = 0$  for  $a > N_f - N_c$ , so this parametrization describes a  $(N_f - N_c)$  dimensional slice of the Higgs branch. To identify a point on the Higgs branch of the form (3.19), it is sufficient to give only the values of the meson field  $M_j^i$ , whose non-vanishing components may be written in the form of an  $N_c \times (N_f - N_c)$  matrix,

$$\hat{M}_i^j = M_i^{j+N_c} \quad i = 1, \dots, N_c, \quad j = 1, \dots, N_f - N_c \quad (3.20)$$

After this small digression, we now return to the vortex worldsheet. So far we have discussed the theory on the vortex only at a special point (2.3), which we may call the origin of the Higgs branch. It is defined in terms of the gauge invariant fields by

$$\text{Origin : } M = \tilde{B} = 0 \quad \text{and} \quad B_{1\dots N_c} = v^{N_c} \quad (3.21)$$

with all other components of  $B$  vanishing. Here we would like to ask how the vortex worldsheet theory responds to motion in the Higgs branch. We will show that sitting on a point in the Higgs branch specified by  $\hat{M}$  induces a gauge invariant superpotential on the vortex string worldsheet,

$$\mathcal{W} \sim \hat{M}_i^j \tilde{\Psi}_j \Psi^i \quad (3.22)$$

This superpotential partially lifts the vortex moduli space. When  $\hat{M}$  is of maximal rank  $(N_f - N_c)$ , the surviving vortex moduli space is  $\mathbf{CP}^{2N_c - N_f - 1}$ . This reduction from  $N_c$  to  $(2N_c - N_f)$  is compatible with the  $A_{2N_c - N_f - 1}$  SCFT we found in the previous section when  $\tilde{m}_j = 0$ . A relationship between the 4d Higgs branch and 2d complex masses is also suggested by the brane picture [23].

To see that (3.22) is correct, we return to the Bogomolnyi equations for the vortex. The equations (3.1) were derived under the assumption that  $\tilde{Q} = 0$ . Relaxing this condition, the full Bogomolnyi equations are given by

$$\mathcal{D}_1 Q_i = i \mathcal{D}_2 Q_i \quad , \quad \mathcal{D}_1 \tilde{Q}^i = i \mathcal{D}_2 \tilde{Q}^i \quad , \quad \frac{1}{e^2} F_{12} = \sum_i (Q_i Q_i^\dagger - \tilde{Q}^{i\dagger} \tilde{Q}^i) - v^2 \quad (3.23)$$

together with the F-term condition

$$\sum_{i=1}^{N_f} Q_i^a \tilde{Q}_b^i = 0 \quad . \quad (3.24)$$

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<sup>3</sup>In the language of [18], the non-baryonic branch *is* lifted by the FI parameter, while the baryonic branch survives, deformed.

When  $\tilde{Q}$  has a vev, the space of solutions to these equations is reduced compared to the case where we can set  $\tilde{Q} = 0$ . The troublesome equation is the second in (3.23). Components of  $\tilde{Q}$  which have an expectation value have no non-trivial solutions to this equation. In the Abelian case this follows from the fact that there is no holomorphic line bundle of negative degree. (It may also be seen through a direct study of the Bogomolnyi equations [33, 34]). Let us see the effect in the non-Abelian case. The collective coordinates  $\psi^a$ , which provide homogeneous coordinates on  $\mathbf{CP}^{N-1}$  tell us how the Abelian gauge potential  $a(x^1, x^2)$  describing a vortex profile is embedded in the non-Abelian gauge group. The dictionary is

$$A_b^a(x^1, x^2) = \psi^a \bar{\psi}_b a(x^1, x^2) \tag{3.25}$$

Writing  $z = x^1 + ix^2$ , the equation for  $\tilde{Q}$  reads

$$(\mathcal{D}_z \tilde{Q}^i)_a = \partial_z \tilde{Q}_a^i + ia_z \bar{\psi}_a \sum_{b=1}^{N_c} \psi^b \tilde{Q}_b^i = 0 \tag{3.26}$$

This can be satisfied trivially by  $\partial \tilde{Q} = 0$  only if the vortex sits inside the  $U(N_c)$  gauge group in such a way that  $\sum_b \psi^b \tilde{Q}_b^i = 0$ . At the point (3.19) on the Higgs branch, this means that  $\psi^a = 0$  for all  $a = 1, \dots, N_c$  such that  $\tilde{q}_a \neq 0$ . In terms of the gauge invariant meson observables, this condition can be re-expressed as

$$\sum_{i=1}^{N_c} \hat{M}_i^j \psi^i = 0 \quad \text{for each } j = 1, \dots, N_f - N_c \tag{3.27}$$

which is indeed a subset of the restrictions that arises from the worldsheet superpotential (3.22):  $\partial W / \partial \tilde{\psi}_j = 0$ . The remaining restrictions arising from the worldsheet superpotential are given by,

$$\frac{\partial \mathcal{W}}{\partial \psi^i} = \sum_j \hat{M}_i^j \tilde{\psi}_j = 0 \quad \text{for each } i = 1, \dots, N_c \tag{3.28}$$

These conditions remove the scaling modes  $\tilde{\psi}_j$  of the vortex string. Let us now see that these too are implied by the four-dimensional equations of motion. As we mentioned in the last section, these scaling modes can be traced to the presence of the excess scalar fields  $Q_{i+N_c}$ ,  $i = 1, \dots, N_f - N_c$ , that do not gain an expectation value. At the origin of the Higgs branch (3.21) these fields have a profile of the form

$$Q_{i+N_c}^a(x^1, x^2) = \psi^a \tilde{\psi}_i \tilde{q}(x^1, x^2; |\tilde{\psi}_i|) \tag{3.29}$$

where  $\tilde{q}$  is the profile of an Abelian scalar field, of the type discussed in [29], and satisfies the boundary conditions  $\tilde{q} \rightarrow 0$  as  $x \rightarrow \infty$ . This condition holds at the origin of the Higgs branch (3.21). However, once we move into the interior of Higgs branch, and the mesonic field  $\hat{M}$  is non-vanishing, these modes fall foul of the F-term condition (3.24). Equation (3.28) imposes the requirement of the F-term on the worldsheet.

In summary, we have shown that the space of solutions to the vortex equations (3.23) and (3.24) at a non-trivial point on the Higgs branch is given by the zero set of the superpotential (3.22). The remaining solutions are 1/2-BPS, requiring that the worldsheet theory has a vacuum preserving  $\mathcal{N} = (2, 2)$  supersymmetry. Happily, for  $N_f < 2N_c$ , it does.



### 3.5 A comment on S-duality and mirror symmetry

So far we have focussed only on asymptotically free theories with  $N_f < 2N_c$ . Here we make some (very) brief remarks about the scale invariant theory with  $N_f = 2N_c$ . The complex gauge coupling  $\tau = 2\pi i/e^2 - \theta$  is an exactly marginal parameter of the theory. The Seiberg-Witten curve is given by [10]

$$y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 + h(q)(h(q) + 2) \prod_{i=1}^{N_f} (x - h(q)m_S - m_i) \quad (3.30)$$

where  $q = e^{2\pi i\tau}$  at weak coupling but may, in general, receive instanton corrections. The definition of the modular function  $h(q)$  can be found in [8, 10], while the singlet mass is defined by  $m_S = \sum_i m_i/N_f$ . The modular properties of the curve imply a  $\Gamma_0(2)$  duality of the field theory, where  $\Gamma_0(2)$  is the subset of  $SL(2, \mathbf{Z})$  matrices with even upper off-diagonal entry.

The corresponding vortex theory is a  $U(1)$  gauge theory with  $N_c$  chiral multiplets of charge +1 and a further  $N_c$  chiral multiplets of charge -1. It is scale invariant, with the complex FI parameter  $t = r + i\theta \equiv -i\tau$  an exactly marginal parameter.

The modular properties of the four-dimensional theory strongly suggest that there is a similar duality group at play in the two dimensional theory, interchanging kinks and elementary excitations. Dualities of this form are, of course, familiar in two dimensions [35] although, to my knowledge, the exact structure of the duality in the present system has not been worked out. Although the duality involves an inversion of the Kähler class of the target space, reminiscent of T-duality, it appears to differ from the mirror symmetry of [36]. It would be interesting to explore this system further.

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### References

- [1] M. Shifman and A. Yung, *Non-Abelian string junctions as confined monopoles*, *Phys. Rev. D* **70** (2004) 045004 [[hep-th/0403149](#)].
- [2] A. Hanany and D. Tong, *Vortex strings and four-dimensional gauge dynamics*, *JHEP* **04** (2004) 066 [[hep-th/0403158](#)].
- [3] A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037 [[hep-th/0306150](#)].
- [4] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nonabelian superconductors: vortices and confinement in  $N = 2$  SQCD*, *Nucl. Phys. B* **673** (2003) 187 [[hep-th/0307287](#)].
- [5] N. Dorey, *The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*, *JHEP* **11** (1998) 005 [[hep-th/9806056](#)].

- [6] N. Dorey, T.J. Hollowood and D. Tong, *The BPS spectra of gauge theories in two and four dimensions*, *JHEP* **05** (1999) 006 [[hep-th/9902134](#)].
- [7] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation and confinement in  $N = 2$  supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **426** (1994) 19 *Erratum ibid.* **B 430** (1994) 485 [[hep-th/9407087](#)].
- [8] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in  $N = 2$  supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484 [[hep-th/9408099](#)].
- [9] A. Hanany and Y. Oz, *On the quantum moduli space of vacua of  $N = 2$  supersymmetric  $SU(n_c)$  gauge theories*, *Nucl. Phys. B* **452** (1995) 283 [[hep-th/9505075](#)].
- [10] P.C. Argyres, M.R. Plesser and A.D. Shapere, *The Coulomb phase of  $N = 2$  supersymmetric QCD*, *Phys. Rev. Lett.* **75** (1995) 1699 [[hep-th/9505100](#)].
- [11] D. Tong, *Monopoles in the Higgs phase*, *Phys. Rev. D* **69** (2004) 065003 [[hep-th/0307302](#)].
- [12] V. Markov, A. Marshakov and A. Yung, *Non-Abelian vortices in  $N = 1^*$  gauge theory*, *Nucl. Phys. B* **709** (2005) 267 [[hep-th/0408235](#)].
- [13] A. Gorsky, M. Shifman and A. Yung, *Non-abelian Meissner effect in Yang-Mills theories at weak coupling*, *Phys. Rev. D* **71** (2005) 045010 [[hep-th/0412082](#)].
- [14] P.C. Argyres and M.R. Douglas, *New phenomena in  $SU(3)$  supersymmetric gauge theory*, *Nucl. Phys. B* **448** (1995) 93 [[hep-th/9505062](#)].
- [15] P.C. Argyres, M. Ronen Plesser, N. Seiberg and E. Witten, *New  $N = 2$  superconformal field theories in four dimensions*, *Nucl. Phys. B* **461** (1996) 71 [[hep-th/9511154](#)].
- [16] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of  $n = 2$  superconformal field theories in 4-dimensions*, *Nucl. Phys. B* **471** (1996) 430 [[hep-th/9603002](#)].
- [17] T. Eguchi and K. Hori,  *$N = 2$  superconformal field theories in 4 dimensions and A-D-E classification*, [hep-th/9607125](#).
- [18] P.C. Argyres, M.R. Plesser and N. Seiberg, *The moduli space of  $N = 2$  SUSY QCD and duality in  $N = 1$  SUSY QCD*, *Nucl. Phys. B* **471** (1996) 159 [[hep-th/9603042](#)].
- [19] D. Tong, *TASI lectures on solitons*, [hep-th/0509216](#).
- [20] A. D’Adda, P. Di Vecchia and M. Lüscher, *Confinement and chiral symmetry breaking in  $CP^{N-1}$  models with quarks*, *Nucl. Phys. B* **152** (1979) 125.
- [21] E. Witten, *Instantons, the quark model and the  $1/N$  expansion*, *Nucl. Phys. B* **149** (1979) 285.
- [22] E. Witten, *Phases of  $N = 2$  theories in two dimensions*, *Nucl. Phys. B* **403** (1993) 159 [[hep-th/9301042](#)].
- [23] A. Hanany and K. Hori, *Branes and  $N = 2$  theories in two dimensions*, *Nucl. Phys. B* **513** (1998) 119 [[hep-th/9707192](#)].
- [24] M. Shifman, A. Vainshtein and R. Zwicky, *Central charge anomalies in 2D sigma models with twisted mass*, [hep-th/0602004](#).
- [25] E.J. Martinec, *Algebraic geometry and effective Lagrangians*, *Phys. Lett. B* **217** (1989) 431.
- [26] C. Vafa and N.P. Warner, *Catastrophes and the classification of conformal theories*, *Phys. Lett. B* **218** (1989) 51.

- [27] E. Witten, *On the Landau-Ginzburg description of  $N = 2$  minimal models*, *Int. J. Mod. Phys. A* **9** (1994) 4783 [[hep-th/9304026](#)].
- [28] A.D. Shapere and C. Vafa, *BPS structure of Argyres-Douglas superconformal theories*, [hep-th/9910182](#).
- [29] A. Achucarro and T. Vachaspati, *Semilocal and electroweak strings*, *Phys. Rept.* **327** (2000) 347 [*Phys. Rept.* **327** (2000) 427] [[hep-ph/9904229](#)].
- [30] R.S. Ward, *Slowly moving lumps in the  $CP^1$  model in  $(2+1)$ -dimensions*, *Phys. Lett. B* **158** (1985) 424.
- [31] R.A. Leese and T.M. Samols, *Interaction of semilocal vortices*, *Nucl. Phys. B* **396** (1993) 639.
- [32] M. Shifman and A. Yung, *Non-Abelian semilocal strings in  $N = 2$  supersymmetric QCD*, *Phys. Rev. D* **73** (2006) 125012 [[hep-th/0603134](#)].
- [33] A.A. Penin, V.A. Rubakov, P.G. Tinyakov and S.V. Troitsky, *What becomes of vortices in theories with flat directions*, *Phys. Lett. B* **389** (1996) 13 [[hep-ph/9609257](#)].
- [34] A. Achucarro, A.C. Davis, M. Pickles and J. Urrestilla, *Vortices in theories with flat directions*, *Phys. Rev. D* **66** (2002) 105013 [[hep-th/0109097](#)].
- [35] S.R. Coleman, *Quantum Sine-Gordon equation as the massive Thirring model*, *Phys. Rev. D* **11** (1975) 2088.
- [36] K. Hori and C. Vafa, *Mirror symmetry*, [hep-th/0002222](#).